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On 2-level Orthogonal Arrays of Odd Index

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ABSTRACT

In this paper we show that the maximum possible number of constraints for a 2-level orthogonal array of odd index with strength t is $t + 1$.

1. INTRODUCTION

Let λ, t ($\geqslant 2$), and m ($\geqslant t$) be positive integers. Let S be a $m \times \lambda 2^t$ matrix of zeros and ones. Let t be a $t \times \lambda 2^t$ submatrix of S. Then S is called a (2-level) orthogonal array if and only if for each choice of t each of the 2^t possible column vectors occurs exactly λ times. We shall call S an (m, λ, t)-array of the index λ strength t, and m constraints. The terminology comes from experimental design. For a recent paper which gives some applications and some of the background on orthogonal arrays, see Seiden and Zemach [1].

Let $m(\lambda, t)$ denote the maximum possible number of constraints in an orthogonal array. This paper is devoted to proving the following result.

THEOREM *If λ is odd and $t \geqslant \lambda + 1$, then $m(\lambda, t) = t + 1$.*

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Section 2 contains an unenlightening proof of the theorem. Section 3 is devoted to an outline of another proof, which shows the origin of the proof in Section 2.

2. Proof of the Theorem

We first show that it is sufficient to prove that $m(\lambda, t) \leqslant t + 1$, by exhibiting explicitly a $(t + 1, \lambda, t)$-array. This is obtained by writing an orthogonal array with t rows, and adding as a $(t + 1)$-st row a parity bit (i.e., an entry in the $(t + 1)$-st row is 1 if the number of 1's in the column above is odd, and 0 otherwise). It is straightforward to check that this is a $(t + 1, \lambda, t)$-array. It was shown in [1] that if λ is odd such an array cannot be extended to a $t + 2$-rowed array. The present result establishes that the structure of $(t + 1, \lambda, t)$ has no bearing on the possibility of extension.

Furthermore, it is sufficient to assume that $t = \lambda + 1$. To see this, suppose that $t > \lambda + 1$, and S is a (m', λ, t)-array with $m' > t + 1$. Let S' be the matrix obtained from S by keeping only those columns starting with $t - (\lambda + 1)$ zeros, and then crossing off the first $t - (\lambda + 1)$ rows. Then S' is a $(m'', \lambda, \lambda + 1)$-array with $m'' > \lambda + 2$, a contradiction.

We now suppose that we have a fixed $(t + 2, \lambda, t)$-array with $t = \lambda + 1$. The remainder of the proof consists of producing a contradiction.

Let a denote the number of columns of the form 0^{t+2}, i.e., $t + 2$ zeros. Let $A(r, k)$ denote the number of columns of the form $0^r 1^k 0^{t+2-r-k}$ with $0 \leqslant r \leqslant t + 1, 1 \leqslant k \leqslant t + 2 - r$.

A complete description of all possible arrays $(t + 1, \lambda, t)$ is given in [1]. It resulted from the following common property of these arrays: *Any two columns differing in an even number of elements must occur the same number of times.*

The sum of the occurrences of any two columns differing in an odd number of elements is equal to λ. As a consequence of this property we obtain the following two recurrence formulas for arrays $(t + 2, \lambda, t)$:

For k even, $2 \leqslant k \leqslant t + 2,\ 0 \leqslant r \leqslant t + 2 - k$.

$$A(r, k) = \lambda - a - A(r, k - 1) - A(r + k - 1, 1). \tag{1}$$

For k odd, $1 \leqslant k < t + 2,\ 0 \leqslant r \leqslant t + 2 - k$.

$$A(r, k) = a + A(r + k - 1, 1) - A(r, k - 1). \tag{2}$$

These two formulas express merely the above property with respect to columns of type a of the array which resulted from a $t+1$-rowed array by adding the $(r+k)$-th row. Equalities (1) and (2) lead to more complex equalities which could not be grasped easily otherwise.

LEMMA 1. *For k odd, $3 \leqslant k < t+2$, $0 \leqslant r \leqslant t+2-k$ we have*

$$\begin{aligned} A(r,k) = {} & -\tfrac{1}{2}(k-1)\lambda + (k-1)a + A(r+k-1,1) \\ & + A(r+k-2,1) + \cdots + A(r,1). \end{aligned} \tag{3}$$

For k even, $2 \leqslant k \leqslant t+2$, $0 \leqslant r \leqslant t+2-k$, we have

$$\begin{aligned} A(r,k) = {} & \tfrac{1}{2}k\lambda - (k-1)a - A(r+k-1,1) \\ & - A(r+k-2,1) - \cdots - A(r,1). \end{aligned} \tag{4}$$

Proof. Using (1) to evaluate $A(r, k-1)$ and substituting the expression into (2), one obtains

$$A(r,k) = -\lambda + 2a + A(r+k-1,1) + A(r+k-2,1) + A(r,k-2).$$

Reiterating this procedure gives equation (4). Analogously one can obtain equation (3). The next two lemmas make use of the conditions of the theorem λ odd and $t = \lambda + 1$ and do not hold in general.

LEMMA 2. *If λ is odd and $t = \lambda + 1$ then $2a + 2 \leqslant t$.*

Proof. It follows from (4) that

$$0 \leqslant A(0, t+2) \leqslant \tfrac{1}{2}(t+2)\lambda - (t+1)a.$$

Consequently

$$(t+1)\,\tfrac{1}{2}(\lambda+1) > \tfrac{1}{2}t(t+1) - 1 = \tfrac{1}{2}(t+2)\lambda \geqslant (t+1)a.$$

Since λ is odd $a \leqslant \tfrac{1}{2}(\lambda - 1)$ and the lemma follows.

LEMMA 3. $A(2a+1, 1) \geqslant \tfrac{1}{2}(t-2a)$.

Proof. Since $A(2a+1, t-2a+1) \geqslant 0$, we find from (3) that

$$(t-2a)a + A(t+1,1) + A(t,1) + \cdots + A(2a+1,1) \geqslant \tfrac{1}{2}(t-2a)\lambda.$$

Since an orthogonal array with its rows reordered is still an orthogonal array, we may assume that

$$A(0,1) \geqslant A(1,1) \geqslant \cdots \geqslant A(t+1,1). \tag{5}$$

Hence,

$$(t - 2a + 1)\, A(2a + 1, 1) \geqslant \tfrac{1}{2}(t - 2a)(\lambda - 2a). \tag{6}$$

Now let c be the integer $\frac{1}{2}(t - 2a)$. The conclusion of the lemma is $A(2a + 1, 1) \geqslant c$. If this were not so, we would have $(t - 2a + 1) \leqslant (2c + 1)(c - 1) < c(2c - 1) = \frac{1}{2}(t - 2a)(\lambda - 2a)$, contradicting (6).

After this series of lemmas, we now return to the proof of the theorem and produce our contradiction. On the one hand, since $A(0, 2a + 2) > 0$, we conclude from (4) that

$$(2a + 1)a + A(2a + 1, 1) + A(2a, 1) + \cdots + A(0, 1) \leqslant \tfrac{1}{2}(2a + 2)\lambda.$$

Making use of (5) this reduces to

$$(2a + 2)\, A(2a + 1, 1) \leqslant (a + 1)\lambda - (2a + 1)a. \tag{7}$$

On the other hand, making use of Lemma 3, we find that

$$\begin{aligned}(2a + 2)\, A(2a + 1, 1) \geqslant (a + 1)(t - 2a) &> -2a^2 + (\lambda - 1)a + \lambda \\ &= (a + 1)\lambda - a(2a + 1),\end{aligned}$$

contradicting (7).

This concludes the proof of the theorem.

3. Source of the Proof

It seems appropriate in this day and age to apologize for explaining where a proof comes from, and we hereby do so.

Think of trying to construct a $(\lambda + 3, \lambda, \lambda + 1)$-array. We start with a $(\lambda + 1, \lambda, \lambda + 1)$-array. We have to decide for each of the 2^t types of columns, how many 00's, 01's, 10's and 11's to add. Of these 2^{t+2} decisions suppose we make the $t + 3$ which we have denoted by a $A(0, 1),\ldots, A(t + 1, 1)$. If we examine the conditions required in order that the resulting matrix be an orthogonal array and make use of the fact that the sum of the number of choices for a given type of column adds to λ, then, by computing in the proper order, we can determine the remaining $2^{t+2} - (t + 3)$ constants in terms of the $t + 3$ we have chosen and λ. Saying that each of these new constants is non-negative then gives us the following set of inequalities which our $t + 3$ choices must satisfy:

$$(2i - 1)a + \sum_{2i} \leqslant i\lambda \qquad (i = 1, 2,\ldots, \tfrac{1}{2}(\lambda + 3)),$$

$$2\, ia + \sum_{2i+1} \geqslant i\lambda \qquad (i = 1, 2,\ldots, \tfrac{1}{2}(\lambda + 1)),$$

where $\sum_j$ denotes the sum of one of the C_j^{t+2} choices of j or the $A(k, 1)$'s. The remainder of the proof consists in showing that for odd λ, and $t + 3$ non-negative integers, this set of inequalities has no solution.

The proof we have given proceeds as follows. By ordering the $A(k, 1)$'s and taking a as a parameter, we have shown that $A(2a + 1, 1)$ must satisfy 2 inconsistent inequalities (in and following Lemma 3). The computed constants which gave rise to these 2 inequalities were determined ($A(2a + 1, t - 2a + 1)$ and $A(0, 2a + 2)$) and the values of just these 2 constants were computed.

Reference

1. E. Seiden and R. Zemach, On Orthogonal Arrays, *Ann. Math. Statist.* **37** (1966), 1355–1370.